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Eigenvalue Solutions for Euler-Bernoulli Beams on Two-Parameter Foundations Using Stodola-Vianello Iteration Method and Polynomial Basis Functions

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ABSTRACT *: The critical buckling load determination of axially compressed simply supported Euler-Bernoulli beams rested on two-parameter foundations (EBBo2PF) of Filonenko-Borodich, Pasternak or Vlasov is important for their analysis and design. This work uses the Stodola-Vianello iteration method to formulate the governing domain equation, and constructs a fourth degree polynomial shape function that satisfies the simply supported ends. By substitution of the derived polynomial basis function in the Stodola-Vianello iterations and use of boundary conditions, the four constants of integration in the algebraic formulation are calculated. The requirement for convergence of the iteration at the first iteration is used to establish the buckling equation, whose roots yield the eigenvalue from which the critical buckling load is derived. The critical buckling load expression is found to depend upon the two beam-foundation parameters* $\beta_1 l^4$ *and* $\beta_2 l^2$ *; and presented in tabulated forms for given values of the foundation parameters. The critical buckling load solution obtained in this study showed insignificant difference of less than 0.075% from the previously reported solution by Anghel and Mares for all values of the foundation parameters. It is further found that when the second foundation parameter* β_2 *vanishes the buckling load becomes identical to the critical buckling load for Euler-Bernoulli beams on Winkler foundations, EBBoWF.*

KEYWORDS *Euler-Bernoulli beam on two-parameter foundation, eigenvalue, critical buckling load, characteristic buckling equation Stodola-Vianello iteration method.*

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I. INTRODUCTION

Several problems in structural and foundation engineering can be analyzed as beam on elastic foundation problems. Such problems are formulated using the theories of beams and elastic foundations. They are complex soil structure interaction problems and certain other engineering problems have analogous equations and can be solved via the beam on elastic foundation approach (Dutta et al, 2021). In such problems, the structural footing beam and the soil are required to act together in supporting the loads (Dutta et al, 2021). The theories of beams have been derived by Euler and independently by Bernoulli, Timoshenko, Levinson (1981), Sayyad and Ghugal (2011), Dahake and Ghugal (2013) and several other researchers.

Similarly, elastic foundation theories have been developed by Winkler, Pasternak, Vlasov, Filonenko and Borodich and Hetenyi (1946). Euler and Bernoulli working independently used orthogonality hypothesis of cross-sectional planes normal to the longitudinal axis of the beam before deformation would retain plane and normal to the longitudinal axis after deformation to develop the classical beam theory now credited to them and called the Euler-Bernoulli beam theory (EBBT) (Emadi et al 2023). The normality assumption implies that transverse shear deformation is neglected, thus limiting the scope of application of the EBBT to thin plates for which the ratio of thickness h to the span l (h/l) is less than 0.050. In reality EBBT has been shown to yield accurate solutions to thin beam problems since shear deformation effects are actually minimal and negligible in their behaviour (Ike 2018a; Emadi et al, 2023; Ike 2021).

Timoshenko beam theory (TBT) has been developed using equilibrium and variational methods by incorporating shear deformation via a relaxation of the assumptions of orthogonality of plane cross-sections. Literature has shown that TBT is good for predicting the behaviours of moderately thick and thick beams (Ike, 2019). TBT considers the deformations to be made of deformations due to bending and deformation due to shear; and it is a first order shear deformation theory.

Mindlin improved the TBT by simplifying the transverse shear correction factors, k_s meant to include the effect of non-constant shear stresses and strains in the cross-sections (Emadi et al, 2023). However, the absence of a clear theoretically/analytically derived expression for k_s remains a major defect of the Mindlin beam formulation. The formulations of stability of moderately thick beams are presented in Ike et al (2019), Onah et al (2020) and Obara (2014).

Abohadima et al (2015) used Recursive differentiation method (RDM) for the analysis of Timoshenko beams on elastic foundation problems under static, vibration and buckling behaviours. The RDM is an efficient analytical method that was proposed by Taha (2014) and found useful for solving linear and non-linear boundary value problems (BVPs). The method uses Taylor series expansion techniques and has been found to be capable of solving complex beam on elastic foundation problems. Taha and Doha (2015) have used the RDM for the free vibration analysis of thin beam on elastic foundation. Similarly study by Taha and Hadima (2015) used the RDM to obtain critical buckling load solutions for non-homogeneous beams resting on elastic foundations.

Papachristou and Sophianopoulos (2013) used Galerkin method for the complex analysis of buckling of axially compressed beams on elastic foundation with considerations for discontinuous (unbounded) contact of the beam and the foundation. The researchers studied cases of both ends pinned and both ends fixed. De-Angel's and Cancellera (2012) also studied the buckling analysis of beam on elastic foundation. Wstawka et al (2019, 2022) presented analytical studies of beams on elastic foundations for symmetrical and unsymmetrical beam sections.

Binesh (2012) used a mesh-free radial point interpolation method (RPIM) to analyze beam on two-parameter elastic foundation. Hassan (2008) investigated the buckling of beam on elastic foundation (BoEF) under different boundary conditions. Arifin et al (2020) developed solutions for the deflection of elastic foundation beams and applied the results to piles.

Hetenyi (1946) applied the equilibrium method to derive the differential equations of beam on elastic foundation (BoEF) and showed that such boundary value problems (BVPs) are solvable using exact method. He used exact mathematical methods to develop critical buckling load solutions to the problems of compressed beam on elastic foundations for various support conditions.

Timoshenko and Gere (1985) and Wang et al (2005) have also studied buckling problems of BoEFs and derived exact critical buckling load solutions to beam on Winkler foundations (BoWF) under Dirichlet boundary conditions. Aristizabal-Ochoa (2013) have also studied the buckling of BoEF problems for different boundary conditions.

Atay and Coskun (2009) investigated the stability problems of BoEF using variational iteration methods (VIMs). They developed expressions for critical buckling loads for BoWFs for prismatic and non-homogeneous cross-sections and for a variety of boundary conditions. Anghel and Mares (2019) studied the method of collocation based on integral formulation of the domain equations and used it for the stability solutions of BoEF. The collocation method sought for the solution to the domain equation only at the collocation points, yielding an approximate solution valid only at the collocation points, and a simplification of the BVP to an algebraic problem. Ike (2018b) has also used a point collocation technique to derive satisfactorily accurate solutions to the BoWF problem thus demonstrating the simplicity and effectiveness of the point collocation method. However, the method requires increased computational rigour as the number of grid points is increased and the size of the resulting matrix equations is increased.

Soltani (2020) used the method of finite elements to develop the buckling matrices for the stability problems of Timoshenko beam resting on elastic foundations. Mama et al (2020) have applied fifth degree polynomial shape functions in the finite element methodology and consequently developed accurate eigenvalue solutions to the stability problem of BoWFs.

Ike (2018a) used the Fourier sine transform method to develop free vibration solutions to harmonically freely vibrating BoWFs for Dirichlet conditions. The sinusoidal kernel functions of the finite sine transformation satisfies the simply supported boundary conditions of the BVP and the integral formulation reduces to an algebraic problem. The exact eigenvalues are obtained by solving the algebraic problem and the natural frequencies determined from the eigenvalues. Ike et al (2018) used Picard's iterative method to accurately solve Euler column stability problems. Ikwueze et al (2018) have used least squares weighted residual method for the elastic buckling solutions of Euler column with fixed-pinned ends.

Ofondu et al (2018) applied the Stodola-Vianello iteration method (SVIM) for the numerical solutions of the buckling of Euler columns. They derived the iteration formula by successive integration method on the buckling differential equation. They found a polynomial based shape function for columns with fixed-pinned ends and applied it in the iteration formula for the SVIM to develop successive iterations for the buckled deflection. They found that accurate eigenvalues are obtained using a few iteration steps.

Dutta et al (2021) derived a first order three noded Euler-Bernoulli beam finite element using fifth order displacement functions for the analysis of thin beam rested on Pasternak foundations. They used a MATLAB computer program on the formulation and obtained results that compared well with previous solutions in literature. They conducted parametric studies for various loadings, boundary conditions and foundation parameters and proved the rapid convergence of their solution.

In recent research works, Ike (2023a, 2023b, 2023c) applied the Stodola-Vianello method to develop exact buckling loads of thin simply supported beams resting on Winkler and Pasternak foundations, respectively. In the research, exact shape functions which satisfied all the boundary conditions were used in the Stodola-Vianello iteration equation to derive exact eigenvalues, from which the buckling loads were derived for any buckling mode. Critical buckling load was found in each case of foundation type at the first buckling mode. Ike (2023c) derived critical buckling load solution of simply supported thin beam on Winkler foundation using polynomial shape function in Stodola-Vianello iteration method. A fourth degree polynomial shape function that satisfies the simply supported thin beam boundary conditions was used to start the iteration process for the next buckling deflection function. The convergence requirement which is the equality of the nth and $(n+1)$th successive deflections was used to establish the eigenvalue upon which the critical buckling load was determined. It was found that one iteration yielded sufficiently accurate buckling solution which differed from the exact solutions by less than 0.13% for all values of the beam on Winkler foundation parameter. Ike (2023d) used the generalized integral transform method for the free vibration analysis of thin beam on Winkler foundation.

In other recent studies, Ike et al (2023a, 2023b) used the Stodola-Vianello iteration method for the critical buckling load analysis of clamped Euler-Bernoulli beam resting respectively on Winkler and two-parameter foundations. In the study, the researchers used polynomials that satisfied the clamped boundary conditions to derive the next successive buckled deflection and obtained eigenvalues that yielded critical buckling loads hence illustrating the effectiveness and accuracy of the Stodola-Vianello iteration method.

However despite its effectiveness and accuracy, the major issue with the adopted method is that the amount of algebraic computational work increases somewhat unmanageably as the number of successive Stodola-Vianello iterations increases.

II. METHODOLOGY

A. Theory

The domain governing equation for the buckling of Euler-Bernoulli beam resting on two-parameter foundation of the Pasternak, Filonenko-Borodich or Vlasov model is given by: (Ike et al, 2023b), (Ike, 2023a).

$$\frac{d^2}{dx^2}\left(E(x)I(x)\frac{d^2u(x)}{dx^2}\right)+P\frac{d^2u(x)}{dx^2}+k_1u(x)-k_2\frac{d^2u(x)}{dx^2}=q(x) \quad (1)$$

where $u(x)$ is the deflection, x is the longitudinal axis, k_1 is the first foundation parameter, P is compressive load, $q(x)$ is distributed load transversely acting on the beam, E is Young's modulus, I is the moment of inertia, k_2 is the second foundation parameter.

Equation (1) represents the general case where the beam is non-prismatic, non-homogeneous and is subjected to distributed transverse load $q(x)$. For cases where the beam is prismatic in cross-section and homogeneous in material property and no transverse load acts, $q(x)$ would vanish and Equation (1) becomes the homogeneous ordinary differential equation (ODE):

$$\frac{d^4u(x)}{dx^4}+(\alpha-\beta_2)\frac{d^2u(x)}{dx^2}+\beta_1u(x)=0 \quad (2)$$

in which $\alpha=\dfrac{P}{EI}$ (3)

$$\beta_1=\frac{k_1}{EI} \quad (4)$$

$$\beta_2 = \frac{k_2}{EI}$$

From Equation (2)

$$\frac{d^4u(x)}{dx^4} = -\left((\alpha - \beta_2)\frac{d^2u(x)}{dx^2} + \beta_1 u(x)\right) \quad (5)$$

Four successive integrations of Equation (5) give the Stodola-Vianello iteration equations. Integrating Equation (5) with respect to x gives:

$$\frac{d^3u(x)}{dx^3} = -(\alpha - \beta_2)\frac{du(x)}{dx} - \beta_1 \int_0^x u(x)dx + c_1 \quad (6)$$

where c_1 is an integration constant.
Integrating again, Equation (6) becomes:

$$\frac{d^2u(x)}{dx^2} = -(\alpha - \beta_2)u(x) - \beta_1 \int_0^x\int_0^x u(x)\,dxdx + c_1 x + c_2 \quad (7)$$

c_2 is another constant of integration.
Integration of Equation (7) gives:

$$\frac{du(x)}{dx} = \theta(x) = -(\alpha - \beta_2)\int_0^x u(x)\,dx - \beta_1 \int_0^x\int_0^x\int_0^x u(x)\,dx\,dx\,dx + \frac{c_1x^2}{2} + c_2x + c_3 \quad (8)$$

wherein $\theta(x)$ is slope of the EB beam's neutral axis, c_3 is a constant of integration.
The fourth successive integration gives:

$$u(x) = -(\alpha - \beta_2)\int_0^x\int_0^x u(x)\,dxdx - \beta_1 \int_0^x\int_0^x\int_0^x\int_0^x u(x)\,dxdxdxdx + \frac{c_1x^3}{6} + \frac{c_2x^2}{2} + c_3x + c_4 \quad (9)$$

where c_4 is the fourth constant of integration.
From Equation (9), the Stodola-Vianello iteration equations become:

$$u''_{n+1}(x) = -(\alpha - \beta_2)u_n(x) - \beta_1 \int_0^x\int_0^x u_n(x)\,dxdx + c_1x + c_2 \quad (10a)$$

$$u_{n+1}(x) = -(\alpha - \beta_2)\int_0^x\int_0^x u_n(x)dxdx - \beta_1 \int_0^x\int_0^x\int_0^x\int_0^x u_n(x)\,dxdxdxdx + \frac{c_1x^3}{6} + \frac{c_2x^2}{2} + c_3x + c_4 \quad (10b)$$

where $u_n(x)$ is the buckling deflection function for the nth iteration, and $u_{n+1}(x)$ is the deflection function for the $(n + 1)$th iteration.
In the method, convergence to the exact buckling eigenvalue is achieved when:

$$u_{n+1}(x) = u_n(x) \quad (11)$$

The buckling criterion shown in Equation (11) is used to determine the buckling load using this method. The criterion in Equation (11) has been modified in this paper, following the method used by Ike (2023a, 2023b) as:

$$\int_0^l u_{n+1}(x)dx = \int_0^l u_n(x)dx \quad (12)$$

The theory of equations justifies the equivalence of Equations (11) and (12).

III. RESULTS AND DISCUSSION

A simply supported EB beam resting on two-parameter foundation as shown in Fig. 1 is considered.

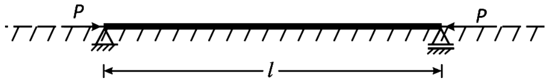

Fig. 1: Buckling of simply supported Euler-Bernoulli (EB) beam on two-parameter foundation

Using fourth degree algebraic polynomials, the buckling shape function is derived from:

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \quad (13)$$

wherein a_0, a_1, a_2, a_3 and a_4 are the constants of the polynomial.

For an E-B beam on the two-parameter foundation simply supported at the left and right ends $(x = 0, x = l)$ respectively, the boundary conditions are:

$$u(x=0) = u''(x=0) = 0 \quad (14a)$$

$$u(x=l) = u''(x=l) = 0 \quad (14b)$$

Using the boundary conditions yield:

$$u(x) = a_4(x^4 - 2lx^3 + l^3 x) = a_4 N(x) \quad (15)$$

where,

$$N(x) = x^4 - 2lx^3 + l^3 x \quad (15a)$$

where $x^4 - 2lx^3 + l^3 x$ is the buckling shape function ($N(x)$) for the simply supported EB beam.

For $u_0(x) = a_4(x^4 - 2lx^3 + l^3 x)$, the next iterations for $u_1''(x)$ and $u_1(x)$ are:

$$u_1''(x) = -(\alpha - \beta_2)a_4(x^4 - 2lx^3 + l^3 x) - \beta_1 \int_0^x \int_0^x a_4(x^4 - 2lx^3 + l^3 x)dxdx + c_1 x + c_2 \quad (16)$$

$$u_1(x) = -(\alpha - \beta_2)\int_0^x \int_0^x a_4(x^4 - 2lx^3 + l^3 x)dxdx - \beta_1 \int_0^x \int_0^x \int_0^x \int_0^x a_4(x^4 - 2lx^3 + l^3 x)dxdxdxdx + \frac{c_1 x^3}{6} + \frac{c_1 x^2}{2} + c_3 x + c_4 \quad (17)$$

Simplifying Equation (16)

$$u_1''(x) = -(\alpha - \beta_2)a_4(x^4 - 2lx^3 + l^3 x) - \beta_1 a_4\left(\frac{x^6}{30} - \frac{lx^5}{10} + \frac{l^3 x^3}{6}\right) + c_1 x + c_2 \quad (18)$$

Using boundary conditions,

$$u_1''(0) = c_2 = 0 \quad (19)$$

$$u_1''(l) = -\beta_1 a_4 (0.1) l^6 + c_1 l = 0 \quad (20)$$

Solving,

$$c_1 = 0.1 a_4 l^5 \beta_1 \quad (21)$$

Then, Equation (17) becomes:

$$u_1(x) = -(\alpha - \beta_2)a_4 \int_0^x \int_0^x (x^4 - 2lx^3 + l^3 x)dxdx - \beta_1 a_4 \int_0^x \int_0^x \int_0^x \int_0^x (x^4 - 2lx^3 + l^3 x)dxdxdxdx - \frac{0.1 a_4 l^5}{6} x^3 \beta_1 + c_3 x + c_4 \quad (22)$$

Integrating Equation (22) gives:

$$u_1(x) = -(\alpha - \beta_2)a_4\left(\frac{x^6}{6} - \frac{lx^5}{10} + \frac{l^3 x^3}{6}\right) - \beta_1 a_4\left(\frac{x^8}{1680} - \frac{lx^7}{420} + \frac{l^3 x^5}{120}\right) + \frac{0.1 a_4 l^5 x^3 \beta_1}{6} + c_3 x + c_4 \quad (23)$$

Using the boundary conditions,

$$u_1(x=0) = c_4 = 0 \quad (24)$$

$$u_1(x=l) = -(\alpha-\beta_2)a_4(0.1l^6) + 0.010119047a_4 l^8 \beta_1 + c_3 l = 0 \quad (25)$$

$$c_3 = (\alpha-\beta_2)0.1a_4 l^5 - 0.010119047a_4 l^7 \beta_1 \quad (26)$$

Hence $u(x)$ for the first iteration is:

$$u_1(x) = -(\alpha-\beta_2)a_4\left(\frac{x^6}{30} - \frac{lx^5}{10} + \frac{l^3x^3}{6}\right) - \beta_1 a_4\left(\frac{x^8}{1680} - \frac{lx^7}{420} + \frac{l^3x^5}{120}\right) + \frac{a_4 l^5 x^3 \beta_1}{60} + (0.1a_4 l^5(\alpha-\beta_2) - 0.010119047a_4 l^7 \beta_1)x \quad (27)$$

The requirement of convergence of $u_0(x)$ and $u_1(x)$ is:

$$\int_0^l u_0(x)dx = \int_0^l u_1(x)dx \quad (28)$$

Hence,

$$\int_0^l a_4(x^4 - 2lx^3 + l^3x)dx = \int_0^l \left\{-(\alpha-\beta_2)a_4\left(\frac{x^6}{30} - \frac{lx^5}{10} + \frac{l^3x^3}{6}\right) - \beta_1 a_4\left(\frac{x^8}{1680} - \frac{lx^7}{420} + \frac{l^3x^5}{120}\right) + \frac{a_4 l^5 x^3 \beta_1}{60} + (0.1a_4 l^5(\alpha-\beta_2) - 0.010119047a_4 l^7 \beta_1)x\,dx\right\} \quad (29)$$

Evaluating the integrals and simplifying,

$$a_4\left(\frac{x^5}{5} - \frac{2lx^4}{4} + \frac{l^3x^2}{2}\right)\Bigg|_0^l = a_4\left[-(\alpha-\beta_2)\left(\frac{x^7}{210} - \frac{lx^6}{60} + \frac{l^3x^4}{24}\right) - \beta_1\left(\frac{x^9}{15120} - \frac{lx^8}{3360} + \frac{l^3x^6}{720}\right) + \frac{l^5x^4}{240}\beta_1 + (\alpha-\beta_2)\frac{l^5x^2}{20} - 5.0595235\times10^{-3} l^7 x^2 \beta_1\right]_0^l \quad (50)$$

Hence, after simplification, it is obtained as follows:

$$\frac{l^5}{5} + 2.050264241\times10^{-3}\beta_1 l^9 = 0.0202389095(\alpha-\beta_2)l^7 \quad (31)$$

Solving for α,

$$(\alpha-\beta_2) = 9.882352941l^{-2} + 0.101307174\beta_1 l^2 \quad (32)$$

Hence,

$$\alpha = \beta_2 + 9.882352941l^{-2} + 0.101307174\beta_1 l^2 \quad (33)$$

$$\alpha = \frac{P}{EI} = \beta_2 + 9.88235294l^{-2} + 0.101307174\beta_1 l^2 \quad (34)$$

$$P = EI(\beta_2 + 9.88235294l^{-2} + 0.101307174\beta_1 l^2) \quad (35)$$

$$P_{cr} = \frac{EI}{l^2}(9.88235294 + \beta_2 l^2 + 0.101307174\beta_1 l^4) \quad (36)$$

$$P_{cr} = \frac{EI}{l^2}K_{cr} \quad (37)$$

$$K_{cr} = 9.88235294 + \beta_2 l^2 + 0.101307174\beta_1 l^4 = \lambda^2 \quad (38)$$

Let $\bar{\beta}_1 = \beta_1 l^4$ (39)

$$\bar{\beta}_2 = \frac{\beta_2 l^2}{\pi^2} \quad (39a)$$

Then

$$K_{cr} = 9.88235294 + \bar{\beta}_2\pi^2 + 0.101307174\bar{\beta}_1 = \lambda^2 \quad (40)$$

So, $K_{cr} = \lambda^2 = \dfrac{P_{cr}l^2}{EI}$ (41)

Hence $\lambda = \sqrt{\dfrac{P_{cr}l^2}{EI}}$ (42)

Values of λ are tabulated in Table 1 for various values of the two-parameter foundation parameters β_1 and β_2.

Table 1: Non-dimensional critical buckling load parameters λ for various values of the two-parameter foundation parameters

		$\bar{\beta}_2 = \beta_2 l^2 = 0$		
$\bar{\beta}_1 = \beta_1 l^4$	Taha (2014)	Anghel and Mares (2019)	Ike (2023a)	Present study
0	3.1415	3.1413	3.141593	3.143621
100	4.4723	4.4721	4.472329	4.473597
		$\bar{\beta}_2 = \beta_2 l^2 = 1$		
$\bar{\beta}_1 = \beta_1 l^4$	Taha (2014)	Anghel and Mares (2019)	Ike (2023a)	Present study
0	4.4428	4.4427	4.442883	4.444317
100	5.4654	5.4653	5.465467	5.466505
		$\bar{\beta}_2 = \beta_2 l^2 = 2.5$		
$\bar{\beta}_1 = \beta_1 l^4$	Taha (2014)	Anghel and Mares (2019)	Ike (2023a)	Present study
0	5.8774	5.8772	5.877382	5.878466
100	6.6840	6.6838	6.683991	6.68484

Table 2: Percentage difference or comparison of present solution for λ and previous solutions

$\bar{\beta}_1 = 0,\ \bar{\beta}_2 = 0$	Present and Taha (2014)	(%) Present and Anghel and Mares (2019)	Present and Ike (2023a)
$\bar{\beta}_1 = 0,\ \bar{\beta}_2 = 0$	0.0675	0.074	0.0646
$\bar{\beta}_1 = 100,\ \bar{\beta}_2 = 0$	0.0290	0.0335	0.0284
$\bar{\beta}_1 = 0,\ \bar{\beta}_2 = 1$	0.0341	0.0364	0.0323
$\bar{\beta}_1 = 100,\ \bar{\beta}_2 = 1$	0.0202	0.022	0.019
$\bar{\beta}_1 = 0,\ \bar{\beta}_2 = 2.5$	0.0181	0.0215	0.0184
$\bar{\beta}_1 = 100,\ \bar{\beta}_2 = 2.5$	0.0126	0.0156	0.0127

V. CONCLUSIONS

This paper has used a fourth degree polynomial shape function constructed to satisfy the boundary conditions at $x = 0,\ \ x = l$ for a simply supported thin beam in the Stodola-Vianello iteration formula for thin beam on two-parameter foundation and determined the approximate critical buckling load solution.
It is concluded as follows:

(i) Stodola-Vianello iteration method simplifies the governing equation of stability of thin beam on two-parameter foundation to an algebraic iteration problem.
(ii) The buckling mode shape function is constructed to satisfy the boundary conditions at the simple supports in the case considered.
(iii) The convergence requirement is used to set up the characteristic buckling equation from which the eigenvalue become the roots.
(iv) The first iteration gives reasonably accurate value of the critical buckling load with differences of less than 0.075% from the exact solution of Anghel and Mares (2019).
(v) The buckling load solution for BoPF become identical with the buckling load solution of BoWF when the second foundation parameter vanishes.
|(vi) The buckling load solution obtained in this study is approximate because the polynomial shape function used in the Stodola-Vianello iteration is approximate, and not exact.

NOMENCLATURE

β_1	Foundation parameter for a one-parameter foundation
β_2	Second parameter for a two-parameter foundation
l	span of a beam resting on an elastic foundation
h	thickness of beam
K_s	transverse shear correction factor
x	longitudinal coordinate axis
$u(x)$	deflection
P	compressive load
$q(x)$	distributed load transversely acting on the beam
E	Young's modulus
I	moment of inertia
α	buckling parameter expressed in terms of P and EI
$\frac{d}{dx}$	first derivative with respect to x
$\frac{d^i}{dx^i}$	ith derivative with respect to x
$\int_0^x (\ )dx$	integral with respect to x
$\int_0^x\int_0^x (\ )dxdx$	two successive integrations with respect to x
$\int_0^x\int_0^x\int_0^x (\ )dxdxdx$	three successive integrations with respect to x
$\int_0^x\int_0^x\int_0^x\int_0^x (\ )dxdxdxdx$	four successive integrations with respect to x
n	buckling mode number
c_1, c_2, c_3, c_4	four constants of integrations
a_1, a_2, a_3, a_4	constants of fourth degree algebraic polynomial function used to express the deflection
$N(x)$	buckling shape function for simply supported beam
P_{cr}	critical buckling load
$\lambda^2 = K_{cr}$	dimensionless critical buckling load parameter
λ	dimensionless critical buckling load parameter expressed in terms of the square root of K_{cr}
$\overline{\beta}_1 = \beta_1 l^4$	dimensionless EB beam on Winkler foundation parameter
$\overline{\beta}_2 = \beta_2 l^2$	dimensionless second parameter for EB beam on two-parameter foundation
%	percent
EBBoPF	Euler-Bernoulli beams rested on two-parameter foundations
BoWF(s)	beam on Winkler foundation(s)
EBBoWF	Euler-Bernoulli beams rested on Winkler foundation
BoEF(s)	beam on elastic foundation(s)
EBBT	Euler-Bernoulli beam theory
BoPF	beam on Pasternak foundation
TBT	Timoshenko beam theory
RDM	Recursive differentiation method
BVP(s)	boundary value problem(s)
RPIM	radial point interpolation method
SVIM	Stodolla-Vianello iteration method
ODE	ordinary differential equation
EB	Euler-Bernoulli
VIM(s)	variational iteration method(s)

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