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Application of Laplace Transform to the Static Analysis of Thin Circular Cylindrical Shell under Internal Hydrostatic Pressure and Ring Force.

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ABSTRACT: *The analysis of a circular cylindrical shell under internal hydrostatic pressure and ring force is carried out in this paper. The governing fourth order differential equation, similar to that of a beam on elastic foundation was adopted from the bending theory of shells. Laplace transform was successfully used to solve the differential equation for the displacements and stresses. The results accurately agreed with those obtained using the classical and initial value methods as well as the method proposed by the Indian Standard IS 3370. The Laplace transform, less tedious and more time saving than its counterparts, proved to be well conditioned for handling the local line force induced by the ring. The introduction of the ring reduced considerably the displacements and stresses in the cylinder. Specifically reduction of 43.75%, 38.85%, 25.10%, 1.65% and 43.75% on the maximum deflection, rotation, bending moment, shear force, and hoop tension respectively was achieved as a result of the introduction of the ring. The optimum location of the ring along the height of the cylindrical shell was also established to be 0.69 times height, measured from the top of cylinder.*

KEYWORDS *Laplace transform, cylindrical shell, hydrostatic pressure, ring force.*

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I. INTRODUCTION

Shells are curved plate structures having a thickness which is very small compared to the other dimensions. Their geometric properties (single or double curvature) make them more efficient than plates in load carrying mechanism Pawel (2005). Single curvature shell structures are commonly used as storage tanks and silos, pressure vessels, submarines, airplanes, chimneys, oil rigs or even lighting columns (Pawel (2005); [Pawel, W. (2005); Lemák, D. and Studnička, J.(2005); Teng, J.G., Zhao, Y. and Lam, L. (2001);Ross, C.T.F., Little, A.P.F. and Adeniyi, K.A. (2005);Wu, T.Y. and Liu, G.R. (2000);Winterstetter, Th.A. and Schmidt, H. (2002); Little, A.P.F., Ross, C.T.F., Short, D. and Brown, G.X. (2008)].

Double curvature shells find their use in the construction of spherical tanks and reservoirs, roofs, stadiums, vehicles and water towers. Shell structures differ in their shape (cylindrical, spherical, parabolic etc…), in the way they are stiffened (laterally, longitudinally or orthogonally stiffened), by the type of load action, by the types of material used (concrete, steel) etc… This great variability and range of shell performance present many difficulties in their design (Lemák, D. and Studnička, J.; 2005).

The objective of this study is to carry out the static analysis of a laterally ring-stiffened circular cylindrical shell in the presence of internal hydrostatic pressure using Laplace transform and to compare the results to those obtained by means of the classical and initial value methods.

Shells support loads through two ways, namely the membrane and bending behaviours. Two distinct shell theories therefore exist: the membrane and the moment theories. A membrane theory is performed under the assumption that a curved surface is incapable of conveying the shear forces and bending moments (Ugural, A.C.; 1999)). A bending theory includes the effects of bending in the analysis. Although, for practical purposes, the membrane stresses are of greater importance than the bending stresses, a bending theory that accounts for the discontinuity effects in geometry (changes in thickness) or boundary conditions (concentrated loading, ring effect) is needed. Indeed these effects cannot be captured by means of the membrane theory alone.

Rings are generally used to stiffen cylindrical shells. They are conventionally used laterally, longitudinally or in both directions. The ring stiffeners prevent elastic buckling from occurring before yielding of the material, further increasing the structural efficiency (Ross, C.T.F.; 1990).

The rigorous shell theories developed in the whole 20th century (Wu, T.Y., Wang, Y.Y. and Liu, G.R.; 2003) often led to sets of differential equations which, in most cases, are solved by means of numerical methods such as finite element, finite difference, differential quadrature methods etc… Even though the tremendous development in the computing technology has permitted very high level of accuracy in these numerical methods, the analytical methods still have their merit in the sense that they provide an insight into the shell problems and an understanding of the shell physical behaviour. They also equip the designer with a basis for evaluating the results of approximate solutions through quantitative comparison. The analytical methods include the classical method, the initial value approach and the Laplace transform, to mention but few. In this study we investigate the use of Laplace transform. The method is widely used in engineering for the solution of differential equations. Its applications include the solution of the differential equation that describes the variation of the charge in a capacitor, the equation of variation of concentration of solids in sewage sludge etc. (Agunwamba, J.C., 2007).

II. THEORY/CALCULATION/METHODOLOGY

A. DIFFERENTIAL EQUATION OF EQUILIBRIUM

The equations of equilibrium of a cylindrical shell according to the bending theory are given as: (Timoshenko, S. and Woinowsky-Krieger, S.; 1959)

$$\frac{\partial^2 u}{\partial x^2}+\frac{1-\mu}{2}\cdot\frac{\partial^2 u}{\partial y^2}+\frac{1+\mu}{2}\cdot\frac{\partial^2 v}{\partial x\partial y}+\frac{u}{R}\cdot\frac{\partial w}{\partial x}+\frac{1-\mu^2}{Eh}\cdot X=0 \quad (1)$$

$$\frac{1+\mu}{2}\cdot\frac{\partial^2 u}{\partial x\partial y}+\frac{1-\mu}{2}\cdot\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial}{\partial y}\left(\frac{w}{R}\right)+\frac{h^2}{12R}\left[\frac{\partial^2}{\partial y^2}\left(\frac{v}{R}\right)-\frac{\partial^3 w}{\partial y^3}-\mu\cdot\frac{\partial^3 w}{\partial x^2\partial y}\right]+\frac{1-\mu^2}{Eh}\cdot Y=0 \quad (2)$$

$$-\frac{\partial^4 w}{\partial x^4}+\frac{\partial}{\partial y}\left[\frac{u}{R}\cdot\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2}{\partial y^2}\left(\frac{v}{R}\right)\right]-\frac{\partial^4 w}{\partial y^4}-2\mu\cdot\frac{\partial^4 w}{\partial x^2\partial y^2}-\frac{12}{Rh^2}\cdot\left[\frac{\partial v}{\partial y}+\frac{w}{R}+\mu\cdot\frac{\partial u}{\partial x}\right]+\frac{12(1-\mu^2)}{Eh^3}\cdot Z=0 \quad (3)$$

where x, y and z, axes at a given point O of the middle surface, are taken in the directions of the axis of the cylinder, the tangent to the circumference, and the normal to the middle surface of the shell respectively; X, Y and Z are the components of the transverse load in x, y and z directions respectively; u, v and w are the displacement components in x, y and z directions respectively; h and R are the thickness and radius of the shell respectively; E and μ are the Young modulus (modulus of elasticity) and the Poisson's ratio of the material respectively. On the assumption that (i) the middle surface of the shell is inextensible in y direction such that v = 0, and (ii) the normal force N_x acting on the transverse section of the shell is neglected, and on the fact that (iii) the forces generated by the hydrostatic and ring force on the vertical cylindrical shell do not depend on y but only on x, the following relationships are obtained:

$$\frac{du}{dx}=u'=-\mu\cdot\frac{w}{R} \quad (4)$$

$$N_\varphi=N=\frac{Ehw}{R} \quad (5)$$

$$S=0 \quad (6)$$

$$M_x=M=-Dw'' \quad (7)$$

$$M_\varphi = \mu M \quad (8)$$

where:

$w'' = \frac{d^2w}{dx^2}$; $D = \frac{Eh^3}{12(1-\mu^2)}$;

N_φ is the hoop tension; S is the membrane shearing force; M_x is the longitudinal bending moment; and M_φ is the transverse bending moment.

Making use of the above conditions (i) and (iii), and noticing that the X and Y components of the loading (Fig. 1) vanish, Equation (3) further becomes:

$$\frac{d^4w}{dx^4} + 4\beta^4 w = \frac{Z}{D} \quad (9)$$

where $\beta^4 = \frac{3(1-\mu^2)}{R^2h^2}$.

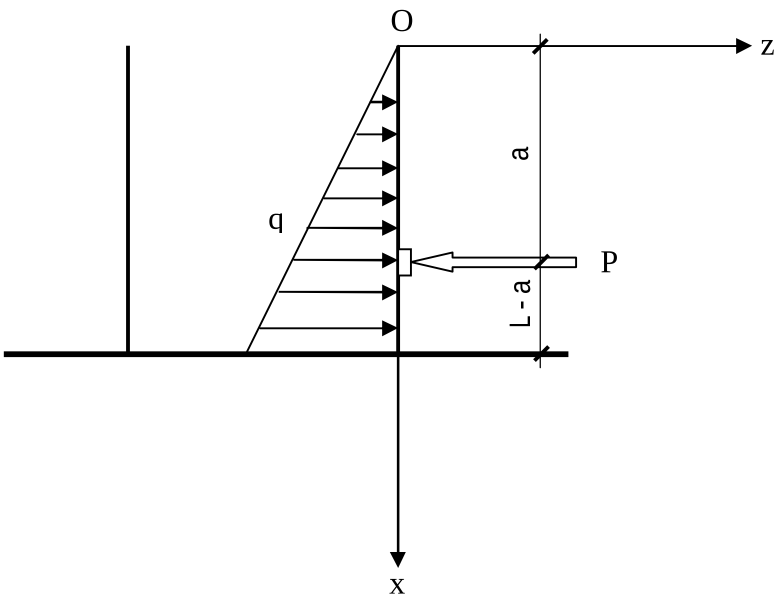

Fig. 1. Section of vertical cylindrical shell under internal hydrostatic pressure and ring force

Equation (9) is only applicable to cylindrical shells subjected to axisymmetric loading, i.e. cases where stresses and displacements are constant along the circumferential section. The Z component of the loading can be expressed as: [13]

$$Z = q - P\delta(x-a) = \gamma x - P\delta(x-a)$$

where q is the internal hydrostatic pressure; P is the ring-induced force applied at the ring location (at distance, a from top of cylinder); γ is specific weight of the contained water; δ(x - a) is known as the Dirac delta function and is defined as follows:

$$\delta(x-a) = \begin{cases} 0 & if \;\; x \neq a \\ 1 & if \;\; x = a \end{cases}$$

Equation (9) can thus be written as:

$$\frac{d^4w}{dx^4} + 4\beta^4 w = \frac{\gamma}{D}x - \frac{P}{D}\delta(x-a) \quad (10)$$

B. Solution

Equation (10) will be solved by means of Laplace transform.

- **Representation of the Differential Equation in Terms of "s" Parameter:**

Laplace transform is next applied to the various terms of Equation (10):

$$\mathcal{L}\left\{\frac{d^4w}{dx^4}\right\} = s^4\overline{w}(s) - s^3w(0) - s^2w'(0) - sw''(0) - w'''(0) \tag{11}$$

$$\mathcal{L}\{4\beta^4 w\} = 4\beta^4\overline{w}(s) \tag{12}$$

$$\mathcal{L}\left\{\frac{\gamma}{D}x\right\} = \frac{\gamma}{D}\cdot\frac{1}{s^2} \tag{13}$$

$$\mathcal{L}\left\{\frac{P}{D}\delta(x-a)\right\} = \frac{P}{D}e^{-as} \tag{14}$$

where $\mathcal{L}\{f(x)\}$ is the Laplace transformation of a function f of x, which is equal to $\overline{f}(s)$; w', w'' and w''' are the 1st, 2nd and 3rd derivatives of w with respect to x respectively.

Consequently the Laplace transform of Equation (10) can be written as:

$$s^4\overline{w}(s) - s^3c_0 - s^2c_1 - sc_2 - c_3 + 4\beta^4\overline{w}(s) = \frac{\gamma}{D}\cdot\frac{1}{s^2} - \frac{P}{D}e^{-as} \tag{15}$$

where $c_0 = w(0)$, $c_1 = w'(0)$, $c_2 = w''(0)$, and $c_3 = w'''(0)$

Making $\overline{w}(s)$ subject of the formula in Equation (15), we obtain:

$$\overline{w}(s) = \frac{s^3c_0 + s^2c_1 + sc_2 + c_3}{s^4 + 4\beta^4} + \frac{\gamma}{D}\cdot\frac{1}{s^2(s^4 + 4\beta^4)} - \frac{P}{D}\cdot\frac{e^{-as}}{s^4 + 4\beta^4} \tag{16}$$

Equation (16) is known as subsidiary equation whose inverse transform yields the solution w(x). Let Equation (16) be written as:

$$\overline{w}(s) = \overline{w_1}(s) + \overline{w_2}(s) + \overline{w_3}(s)$$

where $\overline{w_1}(s) = \frac{s^3c_0 + s^2c_1 + sc_2 + c_3}{s^4 + 4\beta^4}$; $\overline{w_2}(s) = \frac{\gamma}{D}\cdot\frac{1}{s^2(s^4+4\beta^4)}$ and $\overline{w_3}(s) = -\frac{P}{D}\cdot\frac{e^{-as}}{s^4+4\beta^4}$.

It follows that:

$$w(x) = w_1(x) + w_2(x) + w_3(x) \tag{17}$$

where $w_1(x)$, $w_2(x)$ and $w_3(x)$ are the inverse transforms of $\overline{w_1}(s)$, $\overline{w_2}(s)$ and $\overline{w_3}(s)$ respectively.

- **Inverse Transform of $\overline{w_1}(s)$ by Means of Greenwich contour:**

The Laplace transformation of $\overline{w_1}(s)$ can be defined by means of Greenwich contour or complex integral as [11]:

$$w_1(x) = \frac{1}{2\pi i}\oint_C \overline{w_1}(s)e^{sx}$$

where C is the Greenwich contour region and $\oint_C ds$ is the close integral.

By the residue theorem, $w_1(x)$ equals the sum of the residues at the poles of $e^{sx}\overline{w_1}(s)$ within the contour. A residue is obtained by the evaluation of the expression:

$$\frac{1}{(k-1)!}\lim_{s\to s_o}\frac{d^{k-1}}{ds^{k-1}}\left[(s-s_o)^k\bar{f}(s)\right] \tag{19}$$

where k is the order of pole; $\bar{f}(s)$ is the function i.e. $e^{sx}\overline{w_1}(s)$ and s_o is the pole or point of discontinuity. Here the poles are $\sqrt{2i}$; $-\beta\sqrt{2i}$; $\beta\sqrt{-2i}$ and $-\beta\sqrt{-2i}$. Let R_1, R_2, R_3 and R_4 be the residues at the respective poles of $e^{sx}\overline{w_1}(s)$.

$$R_1 = \frac{1}{3!}\lim_{s\to\beta\sqrt{2i}}\frac{d^3}{ds^3}\left[\frac{\left(s-\beta\sqrt{2i}\right)^3(s^3c_0+s^2c_1+sc_2+c_3)e^{sx}}{(s+\beta\sqrt{2i})(s^2+2i\beta^2)}\right]$$
$$= \left(\frac{C_o}{4}+\frac{c_1}{4m}+\frac{c_2}{4m^2}+\frac{c_3}{4m^3}\right)e^{mx} \tag{20}$$

$$R_2 = \frac{1}{3!}\lim_{s\to-\beta\sqrt{2i}}\frac{d^3}{ds^3}\left[\frac{\left(s+\beta\sqrt{2i}\right)^3(s^3c_0+s^2c_1+sc_2+c_3)e^{sx}}{(s-\beta\sqrt{2i})(s^2+2i\beta^2)}\right]$$
$$= \left(\frac{C_o}{4}-\frac{c_1}{4m}+\frac{c_2}{4m^2}-\frac{c_3}{4m^3}\right)e^{-mx} \tag{21}$$

$$R_3 = \frac{1}{3!}\lim_{s\to\beta\sqrt{-2i}}\frac{d^3}{ds^3}\left[\frac{\left(s-\beta\sqrt{-2i}\right)^3(s^3c_0+s^2c_1+sc_2+c_3)e^{sx}}{(s+\beta\sqrt{-2i})(s^2-2i\beta^2)}\right]$$
$$= \left(\frac{C_o}{4}+\frac{c_1}{4n}+\frac{c_2}{4n^2}+\frac{c_3}{4n^3}\right)e^{nx} \tag{22}$$

$$R_4 = \frac{1}{3!}\lim_{s\to-\beta\sqrt{-2i}}\frac{d^3}{ds^3}\left[\frac{\left(s+\beta\sqrt{-2i}\right)^3(s^3c_0+s^2c_1+sc_2+c_3)e^{sx}}{(s-\beta\sqrt{-2i})(s^2-2i\beta^2)}\right]$$
$$= \left(\frac{C_o}{4}-\frac{c_1}{4n}+\frac{c_2}{4n^2}-\frac{c_3}{4n^3}\right)e^{-nx} \tag{23}$$

where $m = \beta\sqrt{2i}$ and $n = \beta\sqrt{-2i}$.

Thus:

$$w_1(x) = \sum_{j=1}^{4} R_j \tag{24}$$

Substitution of Equations (20 - 23) into Equation (24) gives:

$$w_1(x) = \frac{c_o}{4}\left(e^{i\beta x} + e^{-i\beta x}\right)\left(e^{\beta x} + e^{-\beta x}\right) + \frac{c_1}{8\beta}\left(e^{i\beta x} + e^{-i\beta x}\right)\left(e^{\beta x} - e^{-\beta x}\right)$$
$$- \frac{ic_1}{8\beta}\left(e^{i\beta x} - e^{-i\beta x}\right)\left(e^{\beta x} + e^{-\beta x}\right) - \frac{ic_2}{8\beta^2}\left(e^{i\beta x} - e^{-i\beta x}\right)\left(e^{\beta x} - e^{-\beta x}\right)$$
$$- \frac{c_3}{16\beta^3}\left(e^{i\beta x} - e^{-i\beta x}\right)\left(e^{\beta x} + e^{-\beta x}\right)$$
$$- \frac{ic_3}{16\beta^3}\left(e^{i\beta x} + e^{-i\beta x}\right)\left(e^{\beta x} - e^{-\beta x}\right) \quad (25)$$

Owing to $e^u + e^{-u} = 2\cosh u$ and $e^u - e^{-u} = 2\sinh u$, eqn (25) can further be written as:

$$w_1(x) = A_1\varphi_1(\beta x) + A_2\varphi_2(\beta x) + A_3\varphi_3(\beta x) + A_4\varphi_4(\beta x) \quad (26)$$

where $A_1 = c_0$; $A_2 = -\frac{ic_1}{2\beta} - \frac{c_3}{4\beta^3}$; $A_3 = \frac{c_1}{2\beta} - \frac{ic_3}{4\beta^3}$; $A_4 = -\frac{ic_2}{2\beta^2}$;
$\varphi_1(x) = \cos x \cosh x$; $\varphi_2(x) = \sin x \cosh x$; $\varphi_3(x) = \cos x \sinh x$; and
$\varphi_4(x) = \sin x \sinh x$.

- **Laplace Inverse Transform of $\overline{w_2}(s)$ Using Partial Fractions**

$\overline{w_2}(s)$ can be expressed in terms of its partial fraction as:

$$\overline{w_2}(s) = \frac{\gamma}{D}\left[\frac{1}{4\beta^4}\cdot\frac{1}{s^2} + \frac{1}{16\beta^5}\cdot\frac{s}{s^2 + 2\beta s + 2\beta^2} - \frac{1}{16\beta^5}\cdot\frac{s}{s^2 - 2\beta s + 2\beta^2}\right]$$

which, further, gives:

$$\overline{w_2}(s) = \frac{\gamma}{D}\left[\frac{1}{4\beta^4}\cdot\frac{1}{s^2} + \frac{1}{16\beta^5}\cdot\frac{s+\beta}{(s+\beta)^2 + \beta^2} - \frac{1}{16\beta^5}\cdot\frac{\beta}{(s+\beta)^2 + \beta^2} - \frac{1}{16\beta^5}\right.$$
$$\left.\cdot\frac{s-\beta}{(s-\beta)^2 + \beta^2} - \frac{1}{16\beta^5}\cdot\frac{\beta}{(s-\beta)^2 + \beta^2}\right] \quad (27)$$

Taking the inverse transform of Equation (27), we obtain:

$$w_2(x) = \frac{\gamma}{8\beta^5 D}\left[2\beta x + \frac{1}{2}e^{-\beta x}\cos\beta x - \frac{1}{2}e^{-\beta x}\sin\beta x - \frac{1}{2}e^{\beta x}\cos\beta x - \frac{1}{2}e^{\beta x}\sin\beta x\right]$$
$$w_2(x) = \frac{\gamma}{8\beta^5 D}\left[2\beta x - \frac{1}{2}\left(e^{\beta x} - e^{-\beta x}\right)\cos\beta x - \frac{1}{2}\left(e^{\beta x} + e^{-\beta x}\right)\sin\beta x\right]$$
$$w_2(x) = -\frac{\gamma}{8\beta^5 D}\left[-2\beta x + \varphi_2(\beta x) + \varphi_3(\beta x)\right] \quad (28)$$

- **Laplace Inverse Transform of $\overline{w_3}(s)$ Using Partial Fractions**

$\overline{w_3}(s)$, in terms of its partial fractions, is found to be:

$$\overline{w_3}(s) = -\frac{Pe^{-as}}{8D\beta^3}\left[\frac{s+2\beta}{s^2 + 2\beta s + 2\beta^2} - \frac{s-2\beta}{s^2 - 2\beta s + 2\beta^2}\right]$$
$$\overline{w_3}(s) = -\frac{Pe^{-as}}{8\beta^3 D}\left[\frac{s+\beta}{(s+\beta)^2 + \beta^2} + \frac{\beta}{(s+\beta)^2 + \beta^2} - \frac{s-\beta}{(s-\beta)^2 + \beta^2}\right.$$
$$\left.+ \frac{\beta}{(s-\beta)^2 + \beta^2}\right] \quad (29)$$

Taking the inverse transform of Equation (29), we obtain:

$$w_3(x) = -\frac{P}{8\beta^3 D}\left[e^{-\beta(x-a)}\cos(\beta(x-a)) + e^{-\beta(x-a)}\sin(\beta(x-a)) - e^{\beta(x-a)}\cos(\beta(x-a)) + e^{\beta(x-a)}\sin(\beta(x-a))\right]H(x-a)$$

$$w_3(x) = -\frac{P}{8\beta^3 D}\left[-\left(e^{\beta(x-a)} - e^{-\beta(x-a)}\right)\cos(\beta(x-a)) + \left(e^{\beta(x-a)} + e^{-\beta(x-a)}\right)\sin(\beta(x-a))\right]H(x-a)$$

$$w_3(x) = -\frac{P}{4\beta^3 D}[\varphi_2(\beta x - \beta a) - \varphi_3(\beta x - \beta a)]H(x-a) \quad (30)$$

where H(x – a) is the Heaviside unit step function defined as (A. Jeffrey, A.; 2002):

$$H(x-a) = \begin{cases} 0 & if\ x < a \\ 1 & if\ x \geq a \end{cases}$$

Making use of Equations (26, 28, 30) in Equation (17) and recalling that $D = \frac{Eh^3}{12(1-\mu^2)}$ and $\beta^4 = \frac{3(1-\mu^2)}{R^2h^2}$, we obtain:

$$w(x) = A_1\varphi_1(\beta x) + A_2\varphi_2(\beta x) + A_3\varphi_3(\beta x) + A_4\varphi_4(\beta x) - \frac{\gamma R^2}{2Eh\beta}[-2\beta x + \varphi_2(\beta x) + \varphi_3(\beta x)] - \frac{R^2\beta P}{Eh}[\varphi_2(\beta x - \beta a) - \varphi_3(\beta x - \beta a)]H(x-a) \quad (31)$$

Equation (31) represents the expression for the radial deflection. Other displacement and stress components can be found using the following relationships:

Longitudinal rotation, $\theta(x) = w'(x)$;

Longitudinal bending moment, $M(x) = -Dw''(x)$;

Longitudinal shear force, $Q(x) = -Dw'''(x)$;

Hoop tension, $N(x) = \frac{Eh}{R}w(x)$.

It follows that:

$$\theta(x) = \beta(A_2 + A_3)\varphi_1(\beta x) + \beta(A_4 - A_1)\varphi_2(\beta x) + \beta(A_1 + A_4)\varphi_3(\beta x) + \beta(A_2 - A_3)\varphi_4(\beta x) + \frac{\gamma R^2}{Eh}[1 - \varphi_1(\beta x)] - \frac{2R^2\beta^2}{Eh}PH(x-a)\varphi_4(\beta x - \beta a) \quad (32)$$

$$M(x) = -\frac{Eh}{2R^2\beta^2}[A_4\varphi_1(\beta x) - A_3\varphi_2(\beta x) + A_2\varphi_3(\beta x) - A_1\varphi_4(\beta x)] - \frac{\gamma}{4\beta^3}[\varphi_2(\beta x) - \varphi_3(\beta x)] + \frac{P}{2\beta}H(x-a)[\varphi_2(\beta x - \beta a) + \varphi_3(\beta x - \beta a)] \quad (33)$$

$$Q(x) = -\frac{Eh}{2R^2\beta}[(A_2 - A_3)\varphi_1(\beta x) - (A_1 + A_4)\varphi_2(\beta x) + (A_4 - A_1)\varphi_3(\beta x) - (A_2 + A_3)\varphi_4(\beta x)] - \frac{\gamma}{2\beta^2}\varphi_4(\beta x) + PH(x-a)\varphi_1(\beta x - \beta a) \quad (34)$$

$$N(x) = \frac{Eh}{R}[A_1\varphi_1(\beta x) + A_2\varphi_2(\beta x) + A_3\varphi_3(\beta x) + A_4\varphi_4(\beta x)] + \frac{\gamma R}{2\beta}[2\beta x - \varphi_2(\beta x) - \varphi_3(\beta x)] - \beta RPH(x-a)[\varphi_2(\beta x - \beta a) - \varphi_3(\beta x - \beta a)] \quad (35)$$

It should be noted that A_1, A_2, A_3, and A_4 are found using the boundary conditions.

C. Numerical Applications

It is assumed that the top of the cylinder is supported by an elastic ring that has a negligible rigidity against out of plane rotation, while the bottom end is rigidly fixed to the ground. The coordinate system is chosen such that the origin is at the top of the cylinder (see Fig. 1). It follows that:

$$w(0) = M(0) = w(L) = \theta(L) = 0 \quad (36)$$

where L is the height of the cylinder.

Let the tank be made of reinforced concrete and the following values be adopted for the various parameters:

R = 10 m; L = 8 m; h = 0.2 m, γ = 9.81 KN/m^3; E = 25 x 10^6 KN/m^2 and μ = 0.20

Table 1 summarises the stresses and displacements along the height of the tank when only hydrostatic pressure is acting on it. The results are computed from Equations (31 - 35), subject to P = 0 and the boundary conditions enumerated in Equation (36).

Table 1. Forces and displacements in the cylindrical shell under internal hydrostatic pressure by Laplace transform.

X (m)	Deflection (m)	Rotation (rad)	Bending Moment (KNm/m)	Shear Force (KN/m)	Hoop Tension (KN)
0	0	0.000193078	0	-0.0402051	0
0.80	0.000154684	0.00019397	-0.0451496	-0.0880179	77.3422
1.60	0.000311251	0.000198315	-0.158447	-0.19831	155.626
2.40	0.000473902	0.000209739	-0.341304	-0.221658	236.951
3.20	0.000648797	0.000227977	-0.394245	0.208596	324.398
4.00	0.000836118	0.000235429	0.259955	1.6319	418.059
4.80	0.00100885	0.000179067	2.52543	4.14613	504.424
5.33	0.00107679	0.0000639652	5.14438	5.58066	538.396
5.52496	0.00108323	0	6.2528	5.73837	541.615
5.60	0.00108219	-0.0000279551	6.68257	5.70659	541.097
6.40	0.000911624	-0.000422636	9.65614	-0.443004	455.812
7.20	0.000420055	-0.000740247	0.707415	-26.2869	210.028
7.22626	0.00040061	-0.000740787	-3.41061 x 10^{-13}	-27.6061	200.305
8.00	1.38778 x 10^{-17}	0	-39.9695	-79.4167	0

- **The Ring Problem**

Rings are introduced in order to reduce displacements in cylindrical shells. Their introduction will affect the entire stress and displacement distribution in the cylinder. If a single ring should be used, then its ideal location would be the point of maximum deflection due internal hydrostatic pressure. It is assumed here that the width of the ring is negligible compared to the height L of the cylinder, such that the ring action can be idealised as a line (circumferential) force in opposite direction to the hydrostatic pressure. The ring location is found by solving the eqn (37) for a:

$$\frac{dw_{hyd}(x)}{dx} = 0 \tag{37}$$

where $w_{hyd}(x)$ is equal to $w(x)$ subject to P = 0.

A numerical solution of Equation (37) by Newton Raphson method (for example) gives the location of the ring at a = 5.52496 m (measured from the top of the cylinder).

The ring force P is next found by assuming that the ring would induce, at its location, a deflection equal to $w_{hyd}(a)$ but in opposite direction. This is done by solving for P the following equation:

$$w_{ring}(a) = -w_{hyd}(a) \tag{38}$$

where $w_{ring}(x)$ is understood to be equal to $w(x)$ subject to γ = 0.

The solution of Equation (38) yields P = 119.038 KN/m.

The displacements and stresses along the height of shell are given in Table 2. It is worth noting that all computations in this paper were carried out using MATHEMATICA® 5.2 software.

Table 2: Forces and displacements in the cylindrical shell under internal hydrostatic pressure and ring force by Laplace transform.

X (m)	Deflection (m)	Rotation (rad)	Bending Moment (KNm/m)	Shear Force (KN/m)	Hoop Tension (KN)
0	0	0.000193078	0	-0.0402051	0
0.80	0.000174391	0.000220892	-0.153407	0.0339628	87.1955
1.60	0.000352706	0.000221777	0.264219	1.15411	176.353
2.40	0.000517503	0.000177744	1.89298	2.97668	258.752
3.20	0.000608166	0.0000261505	4.82283	3.91817	304.083
3.29082	0.000609367	-4.33681×10^{-19}	5.17351	3.792	304.683
4.00	0.000523554	-0.000252704	6.6514	-0.948996	261.777
4.77416	0.000231914	-0.000452965	1.42109×10^{-14}	-19.1194	115.957
4.80	0.000220213	-0.000452592	-0.505908	-20.0408	110.106
5.33	0.0000205465	-0.00021746	-16.9712	-43.453	10.2732
5.52496	-8.67362×10^{-19}	0.000024404	-26.4394	65.2662	-4.54747×10^{-13}
5.60	5.85806×10^{-6}	0.000128316	-21.6951	61.1801	2.92903
6.40	0.000261007	0.000265765	10.1159	19.1319	130.504
7.20	0.000242178	-0.000306588	9.20791	-22.5769	121.089
8.00	3.81639×10^{-17}	2.77556×10^{-17}	-29.9367	-78.1045	1.54614×10^{-11}

III. DISCUSSION OF RESULTS

The results obtained from the Laplace transform solutions in Tables 1 and 2 are found to agree accurately with those computed from the classical and initial value solutions (Osadebe, N.N. and Adamou, A. (2010) shown in Tables 3, 4 and 5.

Table 3. Forces and displacements in the cylindrical shell under internal hydrostatic pressure by the classical method.

X (m)	Deflection (m)	Rotation (rad)	Bending Moment (KNm/m)	Shear Force (KN/m)	Hoop Tension (KN)
0	0	0.000193078	0	-0.0402051	0
0.80	0.000154684	0.00019397	-0.0451496	-0.0880179	77.3422
1.60	0.000311251	0.000198315	-0.158447	-0.19831	155.626
2.40	0.000473902	0.000209739	-0.341304	-0.221658	236.951
3.20	0.000648797	0.000227977	-0.394245	0.208596	324.398

4.00	0.000836118	0.000235429	0.259955	1.6319	418.059
4.80	0.00100885	0.000179067	2.52543	4.14613	504.424
5.33	0.00107679	0.0000639652	5.14438	5.58066	538.396
5.52496	0.00108323	1.97867×10^{-18}	6.2528	5.73837	541.615
5.60	0.00108219	-0.0000279551	6.68257	5.70659	541.097
6.40	0.000911624	-0.000422636	9.65614	-0.443004	455.812
7.20	0.000420055	-0.000740247	0.707415	-26.2869	210.028
8.00	2.1684×10^{-19}	2.71051×10^{-20}	-39.9695	-79.4167	1.13687×10^{-13}

Table 4. Forces and displacements in the cylindrical shell under internal hydrostatic pressure by initial value method.

X (m)	Deflection (m)	Rotation (rad)	Bending Moment (KNm/m)	Shear Force (KN/m)	Hoop Tension (KN)
0	0	0.000193078	0	-0.0402051	0
0.80	0.000154684	0.00019397	-0.0451496	-0.0880179	77.3422
1.60	0.000311251	0.000198315	-0.158447	-0.19831	155.626
2.40	0.000473902	0.000209739	-0.341304	-0.221658	236.951
3.20	0.000648797	0.000227977	-0.394245	0.208596	324.398
4.00	0.000836118	0.000235429	0.259955	1.6319	418.059
4.80	0.00100885	0.000179067	2.52543	4.14613	504.424
5.33	0.00107679	0.0000639652	5.14438	5.58066	538.396
5.52496	0.00108323	0	6.2528	5.73837	541.615
5.60	0.00108219	-0.0000279551	6.68257	5.70659	541.097
6.40	0.000911624	-0.000422636	9.65614	-0.443004	455.812
7.20	0.000420055	-0.000740247	0.707415	-26.2869	210.028
7.22626	0.00040061	-0.000740787	-2.27374×10^{-13}	-27.6061	200.305
8.00	1.38778×10^{-17}	0	-39.9695	-79.4167	7.27596×10^{-12}

Table 5. Forces and displacements in the cylindrical shell under internal hydrostatic pressure and ring force by initial value method.

X (m)	Deflection (m)	Rotation (rad)	Bending Moment (KNm/m)	Shear Force (KN/m)	Hoop Tension (KN)
0	0	0.000193078	0	-0.0402051	0
0.80	0.000174391	0.000220892	-0.153407	0.0339628	87.1955
1.60	0.000352706	0.000221777	0.264219	1.15411	176.353
2.40	0.000517503	0.000177744	1.89298	2.97668	258.752
3.20	0.000608166	0.0000261505	4.82283	3.91817	304.083
3.29082	0.000609367	2.1684×10^{-18}	5.17351	3.792	304.683
4.00	0.000523554	-0.000252704	6.6514	-0.948996	261.777
4.77416	0.000231914	-0.000452965	1.84741×10^{-14}	-19.1194	115.957
4.80	0.000220213	-0.000452592	-0.505908	-20.0408	110.106
5.33	0.0000205465	-0.00021746	-16.9712	-43.453	10.2732
5.52496	-8.67362×10^{-19}	0.000024404	-26.4394	65.2662	0
5.60	5.85806×10^{-6}	0.000128316	-21.6951	61.1801	2.92903
6.40	0.000261007	0.000265765	10.1159	19.1319	130.504
7.20	0.000242178	-0.000306588	9.20791	-22.5769	121.089
8.00	-3.46945×10^{-18}	-1.73472×10^{-18}	-29.9367	-78.1045	1.36424×10^{-11}

The introduction of the ring modified positively the distribution of forces and displacements along the height of the cylindrical shell. As a matter of fact, a comparison of forces and displacements in Tables 1 and 2 shows a reduction of 43.75%, 38.85%, 25.10%, 1.65% and 43.75% on the maximum deflection, rotation, bending moment, shear force and hoop tension respectively as a result of the introduction of the ring. Percentages are computed here based on the absolute values of the forces and displacements.

The optimum location of the single ring stiffener is found to be at 0.69 L, measured from the top of cylinder.

The case of a cylindrical shell fixed at the bottom end and free at the top end under internal hydrostatic pressure was also looked into maintaining the same parameters. The values of the bending moment and hoop tension along the height of shell using the present method were compared to those obtained by means of the approach of the Indian Standard IS 3370 [16]. Table 6 shows that the differences do not exceed 0.8 KNm/m for the bending moment, and 2.1 KN for the hoop tension, and thus are not excessive for engineering purposes.

Table 6. Comparison of results obtained using the present method and the IS 3370 method for cylindrical shell fixed to the ground and free at the top under internal hydrostatic pressure.

X (m)	Bending Moment (KNm/m)			Hoop Tension (KN)		
	Present Method	IS 3370 Method	Difference	Present Method	IS 3370 Method	Difference
0.0	0	0	0	-0.740702	0.000	0.740702
0.8	-0.0311123	0.0000	0.0311123	77.0796	77.6952	0.6156
1.6	-0.148497	0.0000	0.148497	155.609	156.175	0.566
2.4	-0.337466	-0.502272	-0.164786	236.999	238.579	1.58
3.2	-0.393804	-1.00454	-0.610736	324.436	323.338	-1.098
4.0	0.259388	-0.502272	-0.76166	418.075	416.729	-1.346
4.8	2.52492	2.00909	-0.51583	504.426	503.057	-1.369
5.6	6.68234	6.52954	-0.1528	541.095	539.158	-1.937
6.4	9.65609	9.54317	-0.11292	455.81	456.754	0.944
7.2	0.707449	0.502272	-0.202177	210.027	207.972	-2.055
8.0	-39.9694	-39.6795	0.2899	0	0	0

IV. CONCLUSION

The static analysis of a circular cylindrical shell subjected to internal hydrostatic pressure and a ring force was carried out using the Laplace transform. The forces and displacements for a typical cylindrical shell were computed and compared to those calculated from classical and initial value solutions. The results were found to agree very accurately. The introduction of a circumferential (transverse) ring stiffener considerably reduced the bending moment and hoop tension (which are the most significant items in design of cylindrical shell) due internal hydrostatic pressure. The optimum ring location was also established to be at 0.69 L (L is height of cylinder) measured from the top. Conclusively the Laplace transform proved to be less tedious and more time saving than its counterparts, the classical and initial value approaches, and well-conditioned for handling local forces. Comparison of the results of the present method and those of the method of IS 3370 showed no significant differences.

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